Math 1A: Calculus **Instructor: Alexander Paulin**

Handout: Population Dynamics / Maxima and Minima

Discussions 201, 203 // 2018-10-17

1. Population Dynamics

As discussed in lecture, the population $P(t)$ as a function of time can be modeled by the differential equation

$$\frac{dP}{dt} = kP \tag{1.1}$$

where k is the *relative growth rate*. One of the main flaws of this model is that it does not account for the fact that resources are limited and thus, as a population increases, the competition within the population will increase as well.

We can make a somewhat better model by accounting for this fact (albeit rather simplistically). Basically, rather than have the relative growth rate be a constant k, we will say assume has the form $k(1-P/E)$ for some constants k and E. So the relative growth rate will drop as the population increases.

In this section we will analyze the differential equation obtained by making this adjustment:

$$\frac{dP}{dt} = k\left(1 - \frac{P}{E}\right)P. \tag{1.2}$$

When E is very large (representing "very low competition"), this differential equation looks like equation (1.1).

In the following problems, $P(t)$ will be a solution of the differential equation equation (1.2). (Side comment: such a function is called a logistic function.)

Problem A1 (The significance of E)**.** Make qualitative guesses for the behavior of $P(t)$ depending on the starting population size $P(0)$:

(a) $P(0) = 0$.
(b) $0 < P(0) < E$.
(c) $P(0) = E$.
(d) $P(0) > E$.

Problem A2 (Reducing to a familiar problem)**.** Let $u(t) = \dfrac{E}{P(t)} - 1$. Show that u satisfies a differential equation of the form

$$\frac{du}{dt} = Au$$

where A is a constant, and express A in terms of the constants k and E.

Problem A3. Solve the differential equation from the preceding problem (express your solution in the most general form possible).

Problem A4. Now find $P(t)$. Express your answer as a function of t, using the constants E, $P(0)$, and k.

Problem A5 (Sanity check)**.** Check (using limits if appropriate) that your answer to Problem A4 is consistent with what you described in Problem A1.

Problem A6 (Numerical application)**.** An unnamed island has enough resources to support 1000 inhabitants. Initially, there are 500 inhabitants. After a decade, there are 600 inhabitants. Assuming the above model, approximately how many inhabitants will there be after another decade?

2. Maxima and Minima

Problem B1. Let $f(x) = \sin(x) + x/2$.

(a) Find the maximum value of f on the interval $[\frac{\pi}{2}, 2\pi]$.
(b) Find the minimum value of f on the interval $[\frac{\pi}{2}, 2\pi]$.

Problem B2. Let $g(x) = x^{2/3} - x/3$.

(a) Find the maximum value of g on the interval $[-1, 12]$.
(b) Find the minimum value of g on the interval $[-1, 12]$.